# Perfect Delaunay polytopes and Perfect Inhomogeneous Forms

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**Abstract**

A lattice Delaunay polytope $D$ is called *perfect* if it has the property that there is a unique circumscribing ellipsoid with interior free of lattice points, and with the surface containing only those lattice points that are the vertices of $D$. An inhomogeneous quadratic form is called *perfect* if it is determined by such a circumscribing "empty ellipsoid" uniquely up to a scale factor. Perfect inhomogeneous forms are associated with perfect Delaunay polytopes in much the way that perfect homogeneous forms are associated with perfect point lattices. We have been able to construct some infinite sequences of perfect Delaunay polytopes, one perfect polytope in each successive dimension starting at some initial dimension; we have been able to construct an infinite number of such infinite sequences. Perfect Delaunay polytopes are intimately related to the theory of Delaunay polytopes, and to Voronoi's theory of lattice types.

## 1 Introduction

Consider the lattice $\mathbb{Z}^d$, and a lattice polytope $D$. If $D$ can be circumscribed by an ellipsoid $\mathcal{E} = \partial\{\, \mathbf{x} \in \mathbb{R}^d \,|\, f(\mathbf{x}) \le R^2\}$, where $f$ is a quadratic function in $x_1, \ldots, x_d$, with no $\mathbb{Z}^d$-elements interior to $\{\, \mathbf{x} \in \mathbb{R}^d \ \text{vert} \ f(\mathbf{x}) \le R^2\}$, and with all $\mathbb{Z}^d$-elements on $\mathcal{E}$ being the vertices of $D$, we will say that $D$ is a *Delaunay* polytope with respect to the metric form defined by $\mathcal{E}$; more informally, we will say that $D$ is *Delaunay* if it can be circumscribed by an *empty ellipsoid* $\mathcal{E}$. Typically there is a family of empty ellipsoids that

can be circumscribed about a given Delaunay polytope $D$ but, if there is only one, so that $\mathcal{E}$ is uniquely determined by $D$, we will say that $D$ is a *perfect Delaunay polytope*. Perfect Delaunay polytopes are distinguished in that they cannot fit properly inside other lattice Delaunay polytopes. Perfect Delaunay polytopes are fascinating geometrical objects – examples are the six and seven dimensional Gossett polytopes with 27 and 56 vertices known as $2_{21}$ and $3_{21}$ in Coxeter's notation, which appear in the Delaunay tilings for the quadratic forms corresponding to the root lattices $E_6$ and $E_7$.

We have studied perfect Delaunay polytopes by constructing infinite sequences of them, one perfect Delaunay polytope in each successive dimension, starting at some initial dimension. We have been able to construct an infinite number of infinite sequences of perfect Delaunay polytopes. One of our constructions is the sequence of *G-topes*, $\mathrm{G}^d, d = 6, 7, \ldots$ , with the initial term being the six-dimensional Gossett polytope 221 with 27 vertices; each G-tope is asymmetric with respect to central inversion, and $\mathrm{G}^d$ has $\binom{d+2}{2} - 1$ vertices. Another construction is the sequence of *C-topes*, $\mathrm{C}^d, d = 7, 8, \ldots$ , with initial term the seven-dimensional Gossett polytope $3_{31}$ with 56 vertices; each C-tope is symmetric with respect to central inversion, and $\mathrm{C}^d$ has $2\binom{d+1}{2}$ vertices. Just as the six-dimensional Gossett polytope can be represented as a section of the seven dimensional one, each term $\mathrm{G}^d$ of the asymmetric sequence can be represented as a section of the term $\mathrm{C}^{d+1}$ of the symmetric sequence.

Each perfect Delaunay polytope in the sequence of G-topes uniquely determines a lattice that is an analogue of the root lattice $E_6$, and each term in the sequence of C-topes uniquely determines a lattice that is an analogue of the root lattice $E_7$. The Voronoi and Delaunay tilings for the lattices in these sequences of lattices show many features of the lead terms, namely, the Voronoi and Delaunay tilings for the quadratic forms corresponding to the root lattices $E_6$ and $E_7$.

There is a number of reasons why perfect Delaunay polytopes are fascinating objects for study. First, as mentioned before, they are "perfect" inhomogeneous analogs of perfect lattices. This alone seems to be a natural motivation for a geometer of numbers. If the empty ellipsoid $\mathcal{E}$ circumscribing a perfect Delaunay polytope $D$ with the vertex set $\operatorname{vert} D$ is defined by an equation $f(\mathbf{x}) = 0$, where $f$ is an inhomogeneous quadratic form, then all the coefficients of $f$ are uniquely determined (up to a common scaling factor) from the system $\{f(\mathbf{x}) = 1 \mid \mathbf{x} \in \operatorname{vert} D\}$. Clearly, $\operatorname{vert} D$ is then the

set of all points on which $f$ attains its minimum over $\mathbb{Z}^d$. An analogous property for homogeneous quadratic forms is called *perfection* after Korkin and Zolotareff (1873) (see Martinet (2003) and Conway and Sloane (1988) for modern treatment of perfect homogeneous forms). Naturally, in our context $f$ is called an *inhomogeneous perfect form* and $\mathcal{E}$ is referred to as a *perfect ellipsoid*. The vertices of a perfect Delaunay polytope are analogs of the minimal vectors for a perfect form: the minimal possible number of vertices of a perfect Delaunay polytope is $\frac{n(n+1)}{2} + n$, while the minimal number of shortest vectors of a perfect forms is $\frac{n(n+1)}{2}$.

Perfect Delaunay polytopes were first considered by Erdahl in 1975 in connection with lattice polytopes arising from the quantum mechanics of many electrons. He showed (1975) that there are perfect Delaunay polytopes in one-dimensional lattices, and showed that the Gosset polytope $\mathrm{G}^6 = 2_{21}$ with 27 vertices was a perfect Delaunay polytope in the root lattice $E_6$. He also showed that there were no perfect Delaunay polytopes in dimensions 2, 3, and 4. These results were extended by Erdahl (1992) by showing that the 7-dimensional Gosset polytope $\mathrm{C}^7 = 3_{31}$ with 56 vertices is perfect, and that there are no perfect Delaunay polytopes in lattices with dimension lying between one and six. Erdahl also proved that $[0, 1]$, $\mathrm{G}^6$, and $\mathrm{C}^7$ are the only perfect Delaunay polytopes existing in root lattices. Deza, Grishukhin, and Laurent (1992,1995) found more examples of perfect Delaunay polytopes in dimensions 15, 16, 22, and 23. The first construction of infinite sequences of perfect Delaunay polytopes was first announced at the Conference dedicated to the Seventieth Birthday of Sergei Ryshkov (Erdahl, 2001), and later reported by Rybnikov (2001) and Erdahl and Rybnikov (2002). Perfect Delaunay polytopes have been classified up to dimension 7 – Dutour (2004) proved that $\mathrm{G}^6 = 2_{21}$ is the only Gosset polytope for $d = 6$. It is strongly suspected that the existing lists of seven and eight dimensional perfect Delaunay polytopes are complete (see http://www.liga.ens.fr/~dutour/).

Perfect Delaunay polytopes are important in the study of *strongly regular graphs* or, more generally, *theory of association schemes*. In fact, the Schlafli graph and the Gosset graph can be constructed as 1-skeletons of Gosset polytopes $\mathrm{G}^6$ and $\mathrm{C}^7$, and that is precisely how the Gosset graph has been discovered. A *maximal family of equiangular lines* is a metrical concept, closely related to the combinatorial notion of strongly regular graph (see Deza et al (1992), Deza and Laurent (1997) for details). All Delaunay polytopes found by Deza et al have been derived from such families of lines.

Perfect Delaunay polytopes play an important role in the *L-type reduction theory of Voronoi and Delaunay* for positive (homogeneous) quadratic forms. An L-type domain is the collection of all possible quadratic metric forms that give the same Delaunay tiling $\mathcal{D}$ for $\mathbb{Z}^d$. L-type domains are relatively open polyhedral cones, with boundary cells that are also L-type domains – these conical cells fit together to tile the cone of metrical quadratic forms, which is described in the next section in detail. Simplicial Delaunay tilings label the full dimensional conic "tiles", and all other possible Delaunay tilings label the lower dimensional cones. A significant role for perfect Delaunay polytopes is that they provide labels for a subclass of *edge forms for L-types*. Edges-forms are forms lying on extreme rays of full-dimensional L-type domains, which are labels by simplicial Delaunay tilings. Prior to the discovery of the sequence $\{\mathrm{G}^d\}$ of G-topes by Erdahl and Rybnikov in 2001 only finitely many edge forms were known. All of the infinite sequences of perfect Delaunay polytopes that we report on correspond to infinite sequences of edge forms for L-type domains. The significance of extreme $L$-types is much due to their relation to the structure of Delaunay and Voronoi tilings for lattices. The Delaunay tilings that correspond to edge forms play an important role: *The Delaunay tiling for an L-type domain is the intersection of the Delaunay tilings for edge forms for the L-type.* (Erdahl, 2000) There is a corresponding dual statement on the structure of Voronoi polytopes: *The Voronoi polytope $V_\varphi$ for a form $\varphi$ contained in an L-type domain is a weighted Minkowski sum of linear transforms of Voronoi polytopes for each of the edge forms.* The latter dual result was first established by H.-F. Loesch in his 1990 doctoral dissertation, although it was first published by Ryshkov (1998, 1999), who independently rediscovered Loesh's theorem; this dual result was given a shorter and simpler proof by Erdahl (2000).

Edge forms that are interior to the cone of metric forms are rare in low dimensions. They first occur in dimension 4: $D_4$ has an extreme $L$-type. A good proportion, but not all, of the edge forms appearing in lower dimensions relate either directly or indirectly to perfect Delaunay polytopes. As shown by Dutour and Vallentin (2003) this situation does not persist in higher dimensions: there is an explosion of edge forms in six dimensions, and that only a tiny fraction of these are inherited from perfect Delaunay polytopes.

All of our sequences of perfect Delaunay polytopes determine corresponding sequences of lattices with similar combinatorial properties. Our constructions are first steps of a program to explore the geometry of higher dimensional lattices through infinite sequences of lattices. The infinite sequences

we have constructed are particularly interesting because of the role they play in the structure theory of Delaunay polytopes and Delaunay tilings, which are described in the following section.

# 2 Homogeneous and Inhomogeneous Forms on Lattices

The Voronoi and Delaunay tilings for point lattices are constructed using the Euclidean metric, but are most effectively studied by mapping the lattice onto $\mathbb{Z}^d$, and replacing the Euclidean metric by an equivalent metrical form. For a $d$-dimensional point lattice $\mathbf{\Lambda}$ with a basis $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_d$ this is done as follows. If $\mathbf{v}$ is a lattice vector with coordinates $z_1, z_2, ..., z_d$ relative to this basis, then $\mathbf{v}$ can be written as $\mathbf{v} = \mathbf{Bz}$, where $\mathbf{B} = [\mathbf{b_1}, \mathbf{b_2}, ..., \mathbf{b_d}]$ is the basis matrix. The squared Euclidean length, is given by $|\mathbf{v}|^2 = \mathbf{z}^T\mathbf{B}^T\mathbf{B}\mathbf{z} = \varphi_{\mathbf{B}}(\mathbf{z})$, where $\mathbf{z}$ is the column vector given by $[z_1, z_2, ..., z_d]^T$. This squared length can equally well be interpreted as the squared length of the integer vector $\mathbf{z} \in \mathbb{Z}^d$ relative to the metrical form $\varphi_{\mathbf{B}}$. Therefore, the Voronoi and Delaunay tilings for $\mathbf{\Lambda}$, constructed using the Euclidean metric, can be studied using the corresponding Voronoi and Delaunay tilings for $\mathbb{Z}^d$ constructed using the metrical form $\varphi_{\mathbf{B}}$. Moreover, variation of the Voronoi and Delaunay tilings for $\mathbf{\Lambda}$ in response to variation of the lattice basis $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_d$ can be studied by varying the metrical form $\varphi$ for the fixed lattice $\mathbb{Z}^d$. In the discussion below we will keep the lattice fixed at $\mathbb{Z}^d$, and vary the metrical form $\varphi$. With slight abuse of terminology we call any semidefinite form *metric.*

`Definition:` The *squared length* of a vector $\mathbf{v}$ with respect to a metric form $\varphi$ is called *the norm of* $\mathbf{v}$ *relative to* $\varphi$.

**The inhomogeneous domain of a Delaunay polytope:** The following discussion of the geometry of $\mathcal{P}$ is a summary of some results contained in (Erdahl, 1992). Let $\mathcal{P}$ be the cone of quadratic functions on $\mathbb{R}^d$ defined by:

$$\mathcal{P} = \{\, f \in \mathbb{R}[x_1, \ldots, x_n] \,\big|\, \deg f = 2, \;\; \forall \mathbf{z} \in \mathbb{Z}^d \; f(\mathbf{z}) \geq 0 \,\}.$$

The condition $\forall \mathbf{z} \in \mathbb{Z}^d \quad f(\mathbf{z}) \geq 0$ requires the quadratic part of $f$ to be positive semi-definite, and requires any subset of $\mathbb{R}^d$ where $f$ assumes negative values to be free of $\mathbb{Z}^d$-elements. The real surface determined by the equation

$f(\mathbf{x}) = 0$ might be empty set, it might be a subspace, or it might have the form

$$\mathcal{E}_f = \mathcal{E}_0 \times K,$$

where $\mathcal{E}_0$ is an ellipsoid and $K$ a complementary subspace. The last case is the interesting one - depending on the dimension of $K$, $\mathcal{E}_f$ is either an empty (of lattice points) ellipsoid or an empty cylinder with ellipsoid base.

We denote by $V(f)$ the set $\mathcal{E}_f \cap \mathbb{Z}^d$. In the case where the surface $\mathcal{E}_f$ includes integer points and is an empty ellipsoid, $V(f) = \mathcal{E}_f \cap \mathbb{Z}^d$ is the vertex set for the corresponding Delaunay polytope $D_f = \operatorname{conv} V(f)$. Conversly, if $D$ is a Delaunay polytope in $\mathbb{Z}^d$, there is a circumscribing empty ellipsoid $\mathcal{E}_D$ determined by a function $f_D \in \mathcal{P}$. More precisely, since $D$ is assumed Delaunay there is a metrical form $\varphi_D$, a center $\mathbf{c}$ and radius $R$, so that $f_D(\mathbf{x}) = \varphi_D(\mathbf{x} - \mathbf{c}) - R^2 = 0$ is the equation of a circumscribing empty ellipsoid $\mathcal{E}_D$. Since $f_D$ is non-negative on $\mathbb{Z}^d$, it is an element of $\mathcal{P}$.

In the case where $\mathcal{E}_f$ is an empty cylinder, there will be an infinite number of integer points lying on this surface. When this happens $V(f) = \mathcal{E}_f \cap \mathbb{Z}^d$ is the vertex set for a non-bounded *Delaunay polyhedron* $D_f = \operatorname{conv} V(f)$, which is a cylinder with a base, which is also called a Delaunay polytope. Recall than a convex polyhedron is called a polytope when it is bounded.

`Definition:` Let $D$ be a Delaunay polyhedron. Then, the (inhomogeneous) domain $\mathcal{P}_D$ for $D$ is:

$$\mathcal{P}_D = \{\, f \in \mathcal{P} \mid \ D_f = D \,\}$$

Such domains are relatively open convex cones that partition $\operatorname{int} \mathcal{P}$ (in fact, they partition a larger subset of $\mathcal{P}$, which includes $\operatorname{int} \mathcal{P}$).

The elements $f \in \mathcal{P}_D$ satisfy the homogeneous linear equations $f(\mathbf{z}) = 0$, $\mathbf{z} \in \operatorname{vert} D = D \cap \mathbb{Z}^d$, so the dimensions of domains vary depending upon the rank of this system. When $D$ is a single edge, the rank is one and $\mathcal{P}_D$ is a relatively open facet of the partition with dimension $\dim \mathcal{P} - 1 = \binom{d+2}{2} - 1$. When the rank is equal to $\binom{d+2}{2} - 1$, which is the maximum possible in order that $\mathcal{P}_D$ not be empty, $\mathcal{P}_D$ is an extreme ray of $\mathcal{P}$.

`Definition:` Let $V(p)$ be the set of integer points lying on the boundary of a $d$-dimensional Delaunay polyhedron. A function $p \in \mathcal{P}$ is perfect if and only if the equations $p(\mathbf{z}) = 0$, $\mathbf{z} \in V(p)$ uniquely determine $p$ up to scaling. In this case we will call the subset $V(p) \subset \mathbb{Z}^d$ perfect, and will call $D_p = \operatorname{conv} V(p)$ perfect.

The elements of a 1-dimensional inhomogeneous domain are perfect, and the Delaunay polytopes that determine such domains are perfect. The perfect subsets $V(p)$ must be maximal among the subsets $\{\, V(f) \subset \mathbb{Z}^d \;\big|\; f \in \mathcal{P}\,\}$.

Perfect inhomogeneous forms are analogues of perfect homogeneous forms—both achieve their minimum value on $\mathbb{Z}^d$ sufficiently often that the representations of the minimum determine the form. In the homogeneous case the minimum is taken on the non-zero elements of $\mathbb{Z}^d$; if the minimum value $m_\varphi$ is achieved on the set $V(\varphi)$, then $\varphi$ is uniquely determined by the equations $\varphi(\mathbf{z}) = m_\varphi, \quad \mathbf{z} \in V(\varphi)$. Similarly, in the inhomogeneous case the minimum is taken on $\mathbb{Z}^d$, and the minimum value is zero; if this minimum value is achieved on the set $V(f) \subset \mathbb{Z}^d$, then $f$ is uniquely determined (up to a scale factor) by the equations $f(\mathbf{z}) = 0, \quad \mathbf{z} \in V(f)$.

**Theorem 1** *If $p \in \mathcal{P}$ is perfect, then there is an ellipsoid $\mathcal{E}_0$, where $0 \leq \dim(\mathcal{E}_0) \leq d$, and complementary subspace $K$ so that $\mathcal{E}_p = \mathcal{E}_0 \times K$, and so that $\mathcal{E}_0$ and $K$ satisfy the following arithmetic conditions:*

> $\mathcal{E}_0 \cap \mathbb{Z}^d$ *are the vertices of a perfect Delaunay polytope in* $(\mathrm{aff}\;\; \mathcal{E}_0) \cap \mathbb{Z}^d$;
> $K \cap \mathbb{Z}^d$ *is a sublattice such that* $\mathbb{Z}^d = ((\mathrm{aff}\;\; \mathcal{E}_0) \cap \mathbb{Z}^d) \oplus (K \cap \mathbb{Z}^d)$.

*In this case we have $V(p) = \mathcal{E}_p \cap \mathbb{Z}^d = (\mathcal{E}_0 \cap \mathbb{Z}^d) \bigoplus (K \cap \mathbb{Z}^d)$. Conversely, any ellipsoid $\mathcal{E}_0$ and an affine complementary subspace $K$, satisfying these arithmetic conditions, determine a surface $\mathcal{E}_0 \times K$ for a perfect element $p$ of $\mathcal{P}$ and, therefore, determine a perfect element up to a scale factor.*

By this theorem, if $p$ is perfect, then $D_p$ is either a perfect Delaunay polytope or, in the degenerate case when $\dim K > 0$, a cylinder with a perfect Delaunay polytope as base.

A $\mathbb{Z}^d$-vector $[u_1, \ldots, u_d]$ is called *primitive* if $\mathrm{GCD}(u_1, \ldots, u_d) = 1$. As an example, consider primitive vectors $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$ such that $\mathbf{a} \cdot \mathbf{b} = 1$. Then $\mathcal{E}_0 = \{\mathbf{0}, \mathbf{b}\}$ is an empty ellipsoid in the 1-dimensional lattice $(\mathrm{aff}\, \mathcal{E}_0) \cap \mathbb{Z}^d$, and $K = \mathbf{a}^\perp = \left\{\mathbf{x} \in \mathbb{R}^d \,|\, \mathbf{x} \cdot \mathbf{a} = 0\right\}$ is a complementary subspace. The subset $\{\mathbf{0}, \mathbf{b}\} = \mathcal{E}_0 \cap \mathbb{Z}^d$ is also the vertex set for a Delaunay polytope in $\mathrm{aff}(\mathcal{E}_0 \cap \mathbb{Z}^d)$, and $K$ is defined so that $\mathbb{Z}^d = ((\mathrm{aff}\, \mathcal{E}_0) \cap \mathbb{Z}^d)) \bigoplus (K \cap \mathbb{Z}^d)$. These are the arithmetic conditions stated in the theorem, so that $\{\mathbf{0}, \mathbf{b}\} \times K$ is the surface for a perfect element $p$ of $\mathcal{P}$, which, up to a scale factor, is given by $p(\mathbf{x}) = (\mathbf{a} \cdot \mathbf{x})(\mathbf{a} \cdot \mathbf{x} - 1)$. The preimages of negative real values of $p$ lie between two hyperplanes with equations $\mathbf{a} \cdot \mathbf{x} = 0$ and $\mathbf{a} \cdot \mathbf{x} = 1$, a region which is a degenerate Delaunay polyhedron.

With the exception of the one-dimensional perfect domains, all inhomogeneous domains $\mathcal{P}_D$ have proper faces that are inhomogeneous domains of lesser dimensions. If $D$ is inscribed into another $\mathbb{Z}^d$-polytope $D'$ so that $\operatorname{vert} D' \supset \operatorname{vert} D$, then $\mathcal{P}_{D'} \subset \partial\mathcal{P}_D$.

In the following definition the Delaunay polyhedron $D$ may be of any dimension.

`Definition:` Let $D$ be a lattice Delaunay polyhedron. Then, the *inhomogeneous domain* for $D$ is defined as:

$$\mathcal{P}_D = \{\, f \in \mathcal{P} \mid \operatorname{vert} D = V(f)\,\}$$

If $D$ is a $d$-dimensional Delaunay simplex then $\dim \mathcal{P}_D = \binom{d+1}{2}$, but if $D$ is a perfect Delaunay polyhedron, then $\dim \mathcal{P}_D = 1$.

**Theorem 2** *Let $D$ be a $d$-dimensional Delaunay polyhedron. Then*

$$\mathcal{P}_D = \{ \sum_{\{p \ \mid \ V(p) \supseteq \operatorname{vert} D\}} \omega_p p \ \Big| \ \omega_p \in \mathbb{R}_{>0} \},$$

*where the summation is over all perfect elements $p$ such that* $\operatorname{vert} D \subseteq V(p)$.

In general, not all relatively open faces on the boundary of an inhomogeneous domain $\mathcal{P}_D$ are inhomogeneous domains. However, in the special case where $D$ is $d$-dimensional, all extreme rays are perfect inhomogeneous domains and all relatively open faces are inhomogeneous domains (see Erdahl 1992). In this case an arbitrary element $f \in \mathcal{P}_D$ has the following representation:

$$f = \sum_{\{p \ \mid \ V(p) \supseteq \operatorname{vert} D\}} \omega_p p,$$

where $\omega_p > 0$. The summation is over the *perfect* elements $p$ with the property that $\operatorname{vert} D \subseteq V(p)$.

These last two paragraphs, and our representation theorem for Delaunay polytopes, show the important role played by perfect Delaunay polytopes in the theory.

**The homogeneous domain of a Delaunay tiling:** Voronoi's classification theory for lattices, his *theory of lattice types* (L-types), was formulated using metrical forms and the fixed lattice $\mathbb{Z}^d$. In this theory two lattices are

considered to be the same type if their Delaunay tilings are affinely equivalent. Consider a positive definite quadratic form $\varphi$. Then a lattice polytope $D$ is Delaunay relative to $\varphi$ if it can be circumscribed by so called *empty ellipsoid* $\mathcal{E}$ with equation of the form

$$\varphi(\mathbf{x} - \mathbf{c}) = R^2,$$

where $\mathbf{c} \in \mathbb{R}^d$ and $R \in \mathbb{R}_{>0}$; in addition, $\operatorname{conv} \mathcal{E}$ must have no interior $\mathbb{Z}^d$-elements, and $\operatorname{vert} D$ must be given by $\mathcal{E} \cap \mathbb{Z}^d$. The collection of all such Delaunay polytopes fit together facet-to-facet to tile $\mathbb{R}^d$, a tiling that is uniquely determined by $\varphi$. This is the Delaunay tiling $\mathcal{D}_\varphi$ for $\mathbb{Z}^d$ relative to the metrical form $\varphi$. If a second metrical form $\vartheta$ has Delaunay tiling $\mathcal{D}_\vartheta$, and if $\mathcal{D}_\vartheta$ is identicle to, or affinely equivalent to $\mathcal{D}_\varphi$, then $\varphi$ and $\vartheta$ are metrical forms of the same L-type.

The description we give below requires that certain degenerate metrical forms be admitted into the discussion, namely, those forms $\varphi$ for which $K = \operatorname{Ker} \varphi$ is a rational subspace of $\mathbb{R}^d$. The Delaunay polyhedra for such a form are themselves degenerate–they are cylinders with axis $K$ and Delaunay polytopes as bases. These cylinders fit together to form the degenerate Delaunay tiling $\mathcal{D}_\varphi$. For example, if $\mathbf{a} \in \mathbb{Z}^d$ is primitive, $\varphi(\mathbf{x}) = (\mathbf{a} \cdot \mathbf{x})^2$ is such a form—the kernel $K$ is the solution set for $\varphi(\mathbf{x}) = 0$, and given by $\mathbf{a}^\perp$, which is rational. The Delaunay tiles are infinite slabs, each bounded by a pair of hyperplanes $\mathbf{a} \cdot \mathbf{x} = k, \mathbf{a} \cdot \mathbf{x} = k + 1$, $k \in \mathbb{Z}$. These fit together to tile $\mathbb{R}^d$.

Let $\Phi$ be the cone of metrical forms in $d$ variables, namely, the cone of positive definite quadratic forms and semidefinite quadratic forms with rational kernels. For each metrical form $\varphi \in \Phi$ there is a Delaunay tiling $\mathcal{D}_\varphi$ for $\mathbb{Z}^d$.

`Definition:` If $\mathcal{D}$ is a Delaunay tiling for $\mathbb{Z}^d$, then the following cone of positive definite quadratic forms

$$\Phi_{\mathcal{D}} = \left\{ \varphi \in \Phi^d \;\middle|\; \mathcal{D}_\varphi = \mathcal{D} \right\}$$

is called an L-type domain.

For this definition the Delaunay tilings can be the usual ones, where the tiles are Delaunay polytopes – or they could be degenerate Delaunay tilings where the tiles are cylinders with a common axis $K$.

The relatively open faces of an L-type domain, are L-type domains. If $\mathcal{D}$ is a trangulation, $\Phi_{\mathcal{D}}$ has full dimension $\binom{d+1}{2}$; this is the generic case. If $\mathcal{D}$ is not a triangulation, $\dim \Phi_{\mathcal{D}}$ is less than $\binom{d+1}{2}$, and $\Phi_{\mathcal{D}}$ is a boundary cell of a full-dimensional L-type domain. In the case where $\dim \Phi_{\mathcal{D}} = 1$, the elements $\varphi \in \Phi_{\mathcal{D}}$ are called edge forms as they correspond to extreme rays of full-dimensional L-type domains.

Let $D$ be a polytope in the Delaunay tiling $\mathcal{D}$. Then, if $\pi_\Phi$ is the projection onto the quadratic part, $\Phi_{\mathcal{D}} \subset \pi_\Phi(\mathcal{P}_D)$. Since this containment holds for all Delaunay tiles $D \in \mathcal{D}$, there is the following description of $\Phi_{\mathcal{D}}$ in terms of inhomogeneous domains:

$$\Phi_{\mathcal{D}} = \bigcap_{D \in \mathcal{D}} \pi_\Phi(\mathcal{P}_D),$$

where the intersection is over all $d$-dimensional Delaunay polytopes in $\mathcal{D}$. Since the containment $\Phi_{\mathcal{D}} \subset \pi_\Phi(\mathcal{P}_D)$ also holds for all Delaunay tilings $\mathcal{D}$ that contain $D$, there is the following description of $\pi_\Phi(\mathcal{P}_D)$ in terms of homogeneous domains:

$$\pi_\Phi(\mathcal{P}_D) = \bigsqcup_{\mathcal{D} \ni D} \Phi_{\mathcal{D}},$$

where the disjoint union is over all Delaunay tilings for $\mathbb{Z}^d$ that contain $D$. This holds not only for full-dimensional cells of $\mathcal{P}_D$, but for cells of all dimensions. The last equality shows that $\pi_\Phi(\mathcal{P}_D)$ is tiled by L-type domains. It also establishes the following

**Theorem 3** *If $p \in \mathcal{P}$ is perfect and, therefore, an edge form for an inhomogeneous domain, then $\pi_\Phi(p)$ is an edge form for an L-type domain.*

This result can also be established in a more direct way by appealing to the definition of L-type domain: if $D$ is a perfect Delaunay polytope, or perfect degenerate Delaunay polyhedron, then $\pi_\Phi(\mathcal{P}_D)$ is a one-dimensional L-type domain. By definition, the elements of $\pi_\Phi(\mathcal{P}_D)$ are then edge forms.

The converse of theorem does not hold - there are edge forms for L-type domains that are not inherited from perfect elements in $\mathcal{P}$. Evidence has accumulated recently that the growth of numbers of types of edge forms with dimension is very rapid starting in six dimensions (see Dutour and Vallentin, 2003), but the the growth of perfect inhomogeneous forms is much less rapid - there is some hope that a complete classification can be made through dimension nine, or even through ten.

# 3 Infinite Sequences of Perfect Delaunay polytopes

Consider the following sets of vectors in $\mathbb{R}^d$:

$$D^d_{s,k} = \{\, [1^s, 0^{d-s}] - \frac{s-1}{d-2k}\mathbf{j} \,\} \cup \{\, [1^{s+1}, 0^{d-s-1}] - \frac{s}{d-2k}\mathbf{j} \,\}$$

where $s, k \in \mathbb{N}$ and $\mathbf{j} = [1^d]$ ($1^d$ means the entry 1 is repeated $d$ times, and similarly for $1^s$ and $0^{d-s}$). All permutations of entries are taken so that $|D^d_{s,k}| = \binom{d}{s} + \binom{d}{s+1} = \binom{d+1}{s}$. The following is the main theorem of this paper.

**Theorem 4** *For $d \geq k(2s+1)+1$, where $s \geq 1,\ k \geq 2$, the polytope*

$$P^d_{s,k} = \frac{1}{2}\,\mathrm{conv}\{\ D^d_{s,k} \cup -D^d_{s,k}\ \}.$$

*is a symmetric perfect Delaunay polytope for the affine lattice $\Lambda^d_{s,k} = \mathrm{aff}\, C^d_{s,k}$; the origin is the center of symmetry for $P^d_{s,k}$ and does not belong to $\Lambda^d_{s,k}$. The circumscribing empty ellipsoid can be defined by the equation $\varphi^d_{s,k}(\mathbf{x}) = R^2$, where*

$$\varphi^d_{s,k}(\mathbf{x}) = 4k(d-k(2s+1))|\mathbf{x}|^2 + (d^2 - (4k+2s+1)d + 4k(2s+k))(\mathbf{j}\cdot\mathbf{x})^2.$$

Each pair of positive integers $(s,k)$, for $s \geq 1, k \geq 2$, determines an infinite sequence of symmetric perfect Delaunay polytopes, one in each dimension, with the initial dimension given by $k(2s+1)+1$. For $s=1, k=2$ the infinite sequence is the one described in the opening commentary, i.e., $\mathrm{C}^d, d \geq 7$, where the initial term is the Gosset polytope $3_{21} = \mathrm{C}^7$ with 56 vertices.

The $\binom{d+1}{s}$ diagonal vectors $D^d_{s,k}$ for $P^d_{s,k}$ have the origin as a common mid-point, forming a segment arrangement that generalizes the cross formed by the diagonals of a cross-polytope. Moreover, these $\binom{d+1}{s}$ diagonals are primitive and belong to the same *parity class* for $\Lambda^d_{s,k}$, namely, they are equivalent modulo $2\Lambda^d_{s,k}$. More generally, primitive lattice vectors $\mathbf{u}, \mathbf{v}$ in some lattice $\Lambda$, with mid-points equivalent modulo $\Lambda$, are necessarily equivalent modulo $2\Lambda$ and thus belong to the same parity class. And conversely, the mid-points of lattice vectors $\mathbf{u}, \mathbf{v} \in \Lambda$ belonging to the same parity class are equivalent modulo $\Lambda$. By analogy with the case of cross-polytopes, we call any such

arrangement of segments or vectors a cross. The convex hulls of such crosses often appear as cells in Delaunay tilings – cross polytopes are examples, as are the more spectacular symmetric perfect Delaunay polytopes. There is a criterion due to Voronoi (1908) and Baranovskii (1991) that determines whether these crosses are Delaunay: *Let $\Lambda$ be a lattice, let $\varphi$ be a metrical form, and let $C$ be the convex hull of a cross of primitive vectors belonging to the same parity class. Then $C$ is Delaunay relative to $\varphi$ if and only if the set of diagonal vectors forming the cross is the complete set of vectors of minimal length, relative to $\varphi$, for their parity class.* This is the criterion we have used to establish the Delaunay property for the symmetric perfect Delaunay polytopes $P^d_{s,k}$.

The following result shows that asymmetric perfect Delaunay polytopes can appear as sections of symmetric ones.

**Theorem 5** *For $d \geq 6$ let $\mathbf{u} = [-1^2, 1^{d-1}] \in \mathbb{Z}^{d+1}$. Then*

$$G^d = \text{conv}\{\, \mathbf{v} \in \text{vert}\, P^{d+1}_{1,2} \ \mid\ \mathbf{v} \cdot \mathbf{u} = \frac{1}{2} \,\}$$

*is an asymmetric perfect Delaunay polytope for the affine sublattice $\Lambda^d_{s,k} = \text{aff vert}\, G^d$. The circumscribing empty ellipsoid can be defined as $\{\, \mathbf{x} \in \text{aff}\, G^d \ \mid\ \varphi^{d+1}_{1,2}(\mathbf{x}) = R^2 \,\}$, where*

$$\varphi^{d+1}_{1,2}(\mathbf{x}) = 8(d-5)|\mathbf{x}|^2 + (d^2 - 9d + 22)(\mathbf{j} \cdot \mathbf{x})^2.$$

This is the infinite sequence of G-topes described in the introduction, with Gossett polytope $2_{21} = \mathrm{G}^6$ as the initial term for $d = 6$.

The terms in this sequence have similar combinatorial properties. For example, the lattice vectors running between vertices all lie on the boundary, in all cases. These lattice vectors are either edges of simplicial facets, or diagonals of cross polytope facets–there are two types of facets, simplexes and cross polytopes. The 6-dimensional Gossett polytope has 27 five-dimensional cross-polytopal facets, but for the rest the number is twice the dimension, $2d$. $\mathrm{G}^6$ can be found as a section of $\mathrm{G}^7$, but $\mathrm{G}^8$ and $\mathrm{G}^9$ do not have sections arithmetically equivalent to $\mathrm{G}^6$.

From this theorem we might expect that all asymmetric perfect Delaunay polytopes are obtained from symmetric ones by sectioning, and at our current state of knowledge this appears to be the case. This would be a fortunate turn of events, since the growth of classes of symmeteric perfect Delaunay

| $P$ | $\dim P$ | $\lvert \operatorname{vert} P\rvert$ | *Symmetry* |
|---|---|---|---|
| $P^d_{\kappa,s}$ | $d$ | $2\left(\binom{d}{s}+\binom{d}{s+1}\right)=2\binom{d+1}{s+1}$ | centrally-symmetric |
| $\Upsilon^{d-1}$ | $d-1$ | $\frac{d(d+1)}{2}-1$ | *asymmetric* |

Table 1: Properties of constructed perfect Delaunay polytopes

polytopes with dimension is slower than for all other geometric phenomena associated with point lattices that we know about.

We summarize properties of the constructed polytopes in the following table (in this notation groups of entries separated by semicolumns can only be permuted between themselves).

The following table gives coordinates of the vertices of G-topes, discovered in 2001 by Erdahl and Rybnikov.

| | | |
|---|---|---|
| $[0^d]\times 1$ | $[-1,0^{d-1};-1]\times(d-1)$ | $[1^{d-1};-(d-3)]\times(d-1)$ |
| $[0,1^{d-2};(d-4)]\times(d-1)$ | $[1^2,0^{d-3};1]\times\frac{(d-1)(d-2)}{2}$ | $[1,0^{d-1};0]\times(d-1)$ |

Polytope $P^7_{4,1}$ is affinely equivalent to Gosset's $3_{21} =$ $\mathrm{G}^7$. Polytope $\Upsilon^6$ is affinely equivalent to Gosset's $2_{21} =$ $\mathrm{G}^6$.

# 4 Delaunay Property Part of Main Theorem

We will use the following notation: $\phi_1=\left(\sum_{i=1}^d x_i\right)^2$, $\phi_2(\mathbf{x})=\left|\mathbf{x}-\mathbf{j}\frac{\sum_{i=1}^d x_i}{d}\right|^2$; $\phi_2(\mathbf{x})$ is the squared Euclidean distance from $\mathbf{x}$ to the line $\operatorname{lin}\mathbf{j}$. The following theorem proves the Delaunay property of polytopes $P^d_{s,k}$ asserted in Theorem 4.

**Theorem 6** *Let $s\geq 1, k\geq 2$, and $d\geq k(2s+1)+1$. Let $l^0=[(-1)^{k/2},1^{d-k/2}]$, $\Lambda=< e_1,\dots,e_d,\frac{\mathbf{j}}{d-k}>_{\mathbb{Z}}$, $\Lambda^0=\{\lambda\in\Lambda:\lambda\cdot l^0\equiv 1(\pmod 2)\}$. Then there is a positive definite quadratic form of the type*

$$\phi(\mathbf{x})=\alpha\phi_1(\mathbf{x})+\beta\phi_2(\mathbf{x}), \tag{1}$$

*were $\alpha,\beta>0$, such that the polytope $P^d_{s,k}$ is a Delaunay polytope in the affine lattice $\Lambda^0$ with respect to $\phi$.*

Here $n = d - k$ and $e_1, \dots, e_d$ stand for the canonical basis of $\mathbb{R}^d$.

**Lemma 4.1** *Suppose $\phi(\mathbf{x}) = \alpha\phi_1(\mathbf{x}) + \beta\phi_2(\mathbf{x})$ where $\alpha, \beta > 0$. Then all points $\lambda \in \Lambda^0$ which are closest to 0 with respect to $\phi$, ie.,*

$$\phi(\lambda) = \min\{\phi(u) : u \in \Lambda^0\}$$

*are, up to permutations of components, of the type $[1^l, 0^{d-|l|}] + a\frac{\mathbf{j}}{n}$, where $-\frac{d}{2} \leq l < \frac{d}{2}$, $a \in \mathbb{Z}$. Here $1^l$ for a negative $l$ means $l$ times $-1$. This representation is unique.*

*Proof.* Suppose $\lambda \in \Lambda^0$ is closest to 0 with respect to $\phi$ in $\Lambda^0$. Let $\lambda = a_1e_1 + \cdots + a_de_d + a\frac{\mathbf{j}}{n}$, $a_1, \dots, a_d, a \in \mathbb{Z}$, $m = a_1 + \cdots + a_d$. We have

$$\phi_2(\lambda) = \phi_2\left(\sum_{i=1}^{d} a_ie_i\right) = \left|\sum_{i=1}^{d} a_ie_i - \mathbf{j}\frac{m}{d}\right|^2 = \sum_{i=1}^{d} a_i^2 - \frac{m^2}{d}. \tag{2}$$

The numbers $a_i$ are of two consecutive integer values. Otherwise there would be $a_i$ and $a_j$ such that $a_i - a_j \geq 2$. Consider the vector $\lambda' = a_1e_1 + \cdots + (a_i - 1)e_i + \cdots + (a_j + 1)e_j + \cdots + a_de_d + a\frac{\mathbf{j}}{n}$. We have $\phi_2(\lambda') - \phi_2(\lambda) = (a_i - 1)^2 + (a_j + 1)^2 - a_i^2 - a_j^2 = 2(a_j - a_i) + 2 \leq -2$ and $\mathbf{j} \cdot \lambda' = \mathbf{j} \cdot \lambda$. Since $\lambda' \in \Lambda^0$ and $\phi(\lambda') < \phi(\lambda)$, it follows that the vector $\lambda$ is not closest to 0 which is a contradiction.

Now let $b$ be the smallest of the values of numbers $a_i$. Subtract $b(e_1 + \cdots + e_d)$ from the first part and add an equal value of $bk\frac{\mathbf{j}}{n}$ to the second part of the existing representation of $\lambda$. After a permutation of components, the vector $\lambda$ is equal to $[1^l, 0^{d-l}] + (a + bk)\frac{\mathbf{j}}{n}$ where $0 \leq l < d$. If $l \geq \frac{d}{2}$, again subtract $[1^d]$ from the first summand and add $\mathbf{j}$ to the second summand to get the required representation.

To prove uniqueness, note that the components $\lambda_i$ of the vector $[1^l, 0^{d-|l|}] + a\frac{\mathbf{j}}{n}$ are of at most two values. In vector $[1^l, 0^{d-|l|}]$, either 1s fill the positions of the bigger $\lambda_i$, or $-1$s fill the positions of the smaller $\lambda_i$. Since $-\frac{d}{2} \leq l < \frac{d}{2}$, the choice is unique. □ ∎

**Lemma 4.2** *All vectors in the affine lattice $\Lambda^0$ which are closest to 0 with respect to a form $\phi = \alpha\phi_1(\mathbf{x}) + \beta\phi_2(\mathbf{x})$, $\alpha, \beta > 0$, belong to the set $\{\lambda \in \Lambda^0 : |\lambda \cdot \mathbf{j}| \leq \frac{d}{n}\}$. If $\lambda$ is closest to 0 with respect to $\phi$ and $|\lambda \cdot \mathbf{j}| = \frac{d}{n}$, then $\lambda \in \{\pm\frac{\mathbf{j}}{n}\}$.*

*Proof.* By multiplying both $\alpha$ and $\beta$ by the same positive number, we can assume that the minimal value of $\phi$ on $\Lambda^0$ is 1. Since $\frac{\mathbf{j}}{n} \in \Lambda^0$, $\phi(\frac{\mathbf{j}}{n}) = \alpha(\frac{d}{n})^2 \geq 1$. Let $\lambda \in \mathbb{R}^d$ be a point with $\phi(\lambda) = 1$. Represent $\lambda = \frac{\gamma}{n}\mathbf{j} + u$ where $\gamma \in \mathbb{R}$, $u \cdot \mathbf{j} = 0$. We have $\phi(\lambda) = \alpha\frac{\gamma^2 d^2}{n^2} + \beta|u|^2 = 1$, therefore $\alpha\frac{\gamma^2 d^2}{n^2} \leq 1$. Since $\alpha(\frac{d}{n})^2 \geq 1$, we have proven that $|\gamma| \leq 1$, so $|\lambda \cdot \mathbf{j}| = |\gamma|\frac{d}{n} \leq \frac{d}{n}$.

If the latter inequality holds strictly, then $|\gamma| = 1$ and $\phi(\lambda) = \alpha\frac{d^2}{n^2} + \beta|u|^2 = 1$. Since $\alpha(\frac{d}{n})^2 \geq 1$, we necessarily have $\beta = 0$ so $\lambda = \pm\frac{\mathbf{j}}{n}$. ∎

**Proof of Theorem 6:**

We define a collection of points in $\Lambda^0$ which contains, up to sign and permutation of components, all vectors of the affine lattice $\Lambda^0$ which have the minimal distance to 0 with respect to any quadratic form $\phi$ of the type (1). Then we pick the form $\phi$ so that the vertices of $P^d_{k,s}$ are minimal vectors and other points of $M$ are not. Below is an implementation of this program.

(A) Consider the set

$$M = \left\{ \lambda = [1^l, 0^{d-|l|}] + a\frac{\mathbf{j}}{n} \in \Lambda^0 : 0 \leq \lambda \cdot \mathbf{j} < \frac{d}{n}, -\frac{d}{2} \leq l < \frac{d}{2} \right\} \cup \{\frac{\mathbf{j}}{n}\}. \quad (3)$$

By the two previous lemmas, for every minimal vector $\lambda \in \Lambda^0$ of the quadratic form $\phi$, either $\lambda$ or $-\lambda$, after permutation the components if necessary, belongs to this set.

For $\lambda = [1^l, 0^{d-l}] + a\frac{\mathbf{j}}{n}$, we have

$$\lambda \cdot \mathbf{j} = l + \frac{ad}{n}, \phi_1(\lambda) = \left(l + \frac{ad}{n}\right)^2, \phi_2(\lambda) = |l| - \frac{l^2}{d}, l^0 \cdot \lambda \equiv a + l \pmod 2. \quad (4)$$

From the calculations above, we get

$$M = \left\{ [1^l, 0^{d-|l|}] + a\frac{\mathbf{j}}{n} : l + a \equiv 1(2), 0 \leq lk + ad < d, -\frac{d}{2} \leq l < \frac{d}{2} \right\} \cup \{\frac{\mathbf{j}}{n}\}. \quad (5)$$

We define a mapping from $\mathbb{R}^d$ to $\mathbb{R}^2$ by $\phi_{1,2}(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}))$. We will call the image of $M$ the *diagram.* Lines parallel to $\phi_1$-axis in $\mathbb{R}^2$ will be called horizontal.

(B) For $\lambda_1, \lambda_2 \in M$, points $\phi_{1,2}(\lambda_1), \phi_{1,2}(\lambda_2)$ belong to the same horizontal line if and only if $\lambda_1 = \pm\lambda_2$. If $\lambda$ and $-\lambda$ both belong to $M$, then $\lambda \cdot \mathbf{j} = 0$.

To prove that, take $\lambda = [1^l, 0^{d-|l|}] + a\frac{\mathbf{j}}{n} \in M$. If $l \neq 0$, then the condition $0 \leq lk + ad < d$ uniquely defines $a$ from a given value of $l$, and if $l = 0$, then $a = 1$. Therefore $l$ uniquely defines $a$.

The function $l \to |l| - \frac{l^2}{d}$ is even and increasing on $[0, \frac{d}{2}]$ so two different points of the diagram may belong to the same horizontal line if and only if their preimages are $\lambda_1 = [1^l, 0^{d-|l|}] + a_1\frac{\mathbf{j}}{n}$ and $\lambda_2 = [1^{-l}, 0^{d-|l|}] + a_2\frac{\mathbf{j}}{n}$ for some $l$,$a_1$,$a_2$. If $l \neq 0$, then $0 \leq lk + a_1 d < d$, $0 \leq -lk + a_2 d < d$. Adding these inequalities, we get $0 \leq (a_1 + a_2)d < 2d$, so $a_1 + a_2 \in \{0, 1\}$. If $a_1 + a_2 = 0$, then $\lambda_1 = -\lambda_2$ so $\phi_{1,2}(\lambda_1) = \phi_{1,2}(\lambda_2)$. If $a_1 + a_2 = 1$, then the numbers $l + a_1$ and $-l + a_2$ have different parity which contradicts conditions $l + a_1 \equiv -l + a_2 \equiv 1(2)$. If $l = 0$, then $a_1 = a_2 = 1$. This proves our claim.

(C) Our method of constructing perfect Delaunay polytopes can be summarised as follows. We note that each line $\alpha x_1 + \beta x_2 = 1$, where $\alpha, \beta > 0$, which supports the edge of the convex hull of the diagram, gives rise to a quadratic form $\phi(\mathbf{x}) = \alpha\phi_1(\mathbf{x}) + \beta\phi_2(\mathbf{x})$ such that the ellipsoid $\phi(\mathbf{x}) = 1$ circumscribes the points of the affine lattice $\Lambda^0$ which fall into the endpoints of the edge and has no lattice points inside.

(D) First consider case $d = 7$, $k = 4$, $s = 1$. The diagram for these values of parameters is shown in figure 4. We see that the line $\frac{3}{7}x_1 + \frac{2}{3}x_2 = 1$ passes through points $\phi_{1,2}(v^7_{4,1}) = \phi_{1,2}([1, 0^6])$ and $\phi_{1,2}(v^7_{4,2}) = \phi_{1,2}([-1^2, 0^5] + \frac{\mathbf{j}}{3})$, and all other points of the diagram are contained in the open half-plane $\frac{3}{7}x_1 + \frac{2}{3}x_2 > 1$. This means that polytope $P^7_{4,1}$ is a Delaunay polytope with respect to quadratic form $\frac{3}{7}\phi_1(\mathbf{x}) + \frac{2}{3}\phi_2(\mathbf{x}) = \frac{2}{3}\mathbf{x}\cdot\mathbf{x} + \frac{1}{3}(\mathbf{j}\cdot\mathbf{x})^2$.

(E) Now suppose that $d \geq 8$, $4 \leq k \leq d/2$, $k \equiv 0(2)$ and $1 \leq s \leq \frac{d}{k} - 1$. Suppose $\lambda = [1^l, 0^{d-l}] + a\frac{\mathbf{j}}{n} \in M$ and $0 \leq \phi_2(\lambda) \leq \frac{d}{k} - \frac{d}{k^2}$ (or, equivalently, $|l| \leq \frac{d}{k}$). If $|l| < \frac{d}{k}$, then $l \geq 0$ and

$$\lambda = [1^l, 0^{d-l}] - (l-1)\frac{\mathbf{j}}{n}. \tag{6}$$

If $|l| = \frac{d}{k}$ (this implies that $\frac{d}{k}$ is an integer number), then

$$\pm\lambda = [1^{\frac{d}{k}}, 0^{d-\frac{d}{k}}] - (\frac{d}{k} - 1)\frac{\mathbf{j}}{n}. \tag{7}$$

To prove this, first check that these points belong to set $M$, and then use statement in paragraph (B) that there is at most two points $\lambda = [1^l, 0^{d-|l|}] +$

$a\frac{\mathbf{j}}{n} \in M$ with a given value of $|l|$, and there are two if and only if $lk + ad = 0$, in which case the points are $\pm\lambda$.

We'll use notation $v_{k,s}^d = [1^l, 0^{d-l}] - (l-1)\frac{\mathbf{j}}{n}$ for $0 \le s \le \frac{d}{k}$. We have

$$\phi_1(v_{k,s}^d) = \left(s + (1-s)\frac{d}{n}\right)^2, \phi_2(v_{k,s}^d) = s - \frac{s^2}{d}. \tag{8}$$

All points $\phi_{1,2}(v_{k,s}^d)$ therefore belong to the curve

$$t \to \left(\left(t + (1-t)\frac{d}{n}\right)^2, t - \frac{t^2}{d}\right) \tag{9}$$

which touches the vertical axis in the point $(0, \frac{d}{k} - \frac{d}{k^2})$ when $t = \frac{d}{k}$. The curve is drawn in dashed line in figure 4. The portion of the curve for $0 \le t \le \frac{d}{k}$ is the graph of a convex function.

Therefore, for each $0 \le s \le \frac{d}{k} - 1$ we can find a line with equation $\alpha x_1 + \beta x_2 = 1$ which passes through points $\phi_{1,2}(v_{k,s}^d)$, $\phi_{1,2}(v_{k,s+1}^d)$, supports the convex hull of the diagram and does not contain points $\phi_{1,2}(\lambda)$ for $\lambda \in M$, $\lambda \ne \pm v_{k,s}^d, \pm v_{k,s+1}^d$. Quadratic form $\phi(\mathbf{x}) = \alpha\phi_1(\mathbf{x}) + \beta\phi_2(\mathbf{x})$ defines an empty ellipsoid with center 0 which contains vertices of $P_{k,s}^d$ on its boundary and does not contain any other lattice points.

We have proven that $P_{k,s}^d$ is a Delaunay polytope in affine lattice $\Lambda_0$ with respect to a unique quadratic form $\phi = \phi_{k,s}^d$ for $d \ge 7$. Explicit formula (4) for $\phi_{k,s}^d$ is established by a trivial calculation.

□

Example of the diagram for $d = 19$, $k = 6$ is shown in figure 4 (right-hand image). Note that not all points belong to the curve (9).

# 5 Perfection Property Part of Main Theorem

**Theorem 7** *Let $d$, $k$ and $s$ be as in Theorem 4. Then, there can be at most one positive definite quadratic form $\phi$ whose ellipsoid $\phi(\mathbf{x} - \mathbf{c}) = 1$ circumscribes the polytope $P_{k,s}^d$ for some $\mathbf{c} \in \mathbb{R}^d$.*

*Proof.* As before, $n = d - k$. Since for any given positive definite quadratic form the center of the its ellipsoid which circumscribes $P_{k,s}^d$ is defined uniquely,

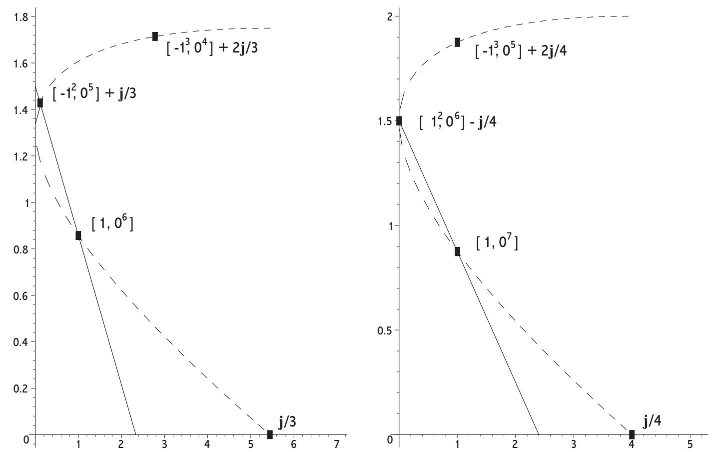


Figure 1: Diagrams for $d = 7, 8$, $\varkappa = 4$

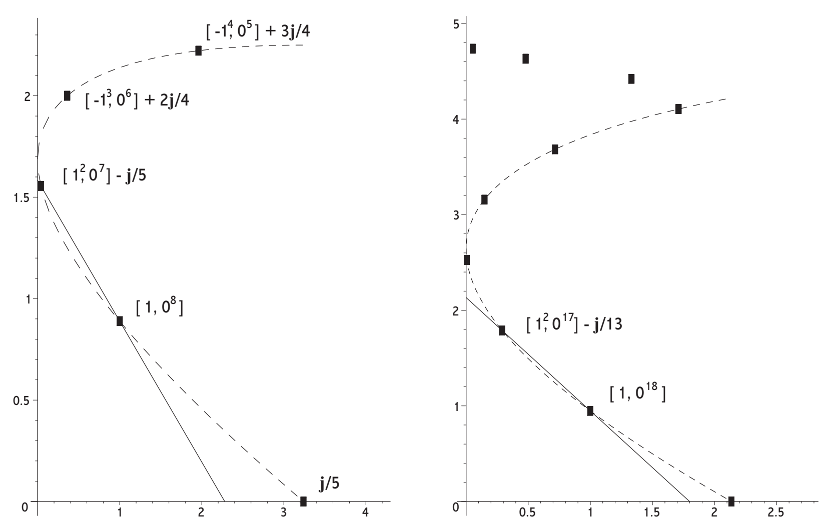


Figure 2: Diagrams for $d = 9$, $\varkappa = 4$ and $d = 19$, $\varkappa = 6$

and since the polytope $P^d_{k,s}$ has 0 as the center of symmetry, 0 is the center of the circumscribing ellipsoid. Therefore, if $\phi$ and $\psi$ are two positive definite quadratic forms which circumscribe $P^d_{k,s}$ by ellipsoids of radius 1, then the ellipsoids are given by equations $\phi(\mathbf{x}) = 1$, $\psi(\mathbf{x}) = 1$. Hence $(\phi - \psi)|_{V^d_{k,s} \cup V^d_{k,s+1}} = 0$. We consider an arbitrary quadratic form $f$ such that $f|_{V^d_{k,s} \cup V^d_{k,s+1}} = 0$ and prove that $f = 0$.

We will use the following symmetrization technique. Suppose that $G$ is a subgroup of the group $S_d$ of permutations on $d$ elements $I_d = \{1, \ldots, d\}$. The symmetrization $F$ of the form $f$ by $G$ is defined as

$$F(x_1, \ldots, x_d) = \sum_{\sigma \in G} f(x_{\sigma 1}, \ldots, x_{\sigma d}). \tag{10}$$

Since polytope $P^d_{k,s}$ is invariant under transformations of $\mathbb{R}^d$ defined by permutations of coordinates, all components in the above sum are 0 on the set of vertices of $P^d_{k,s}$. Therefore $F|_{V^d_{k,s} \cup V^d_{k,s+1}} = 0$.

Firstly we prove that if a form $f$ satisfies the condition $f|_{V^d_{k,s} \cup V^d_{k,s+1}} = 0$, then the sum of the diagonal elements of $f$, and the sum of off-diagonal elements are 0. We symmetrize $f$ by the group $S_d$ and get a form $F$ with diagonal coefficients proportional to the sum of diagonal coefficients of $f$, and non-diagonal coefficients proportional to the sum of non-diagonal coefficients of $f$. Therefore $F$ can be written as $F(\mathbf{x}) = \alpha\phi_1(\mathbf{x}) + \beta\phi_2(\mathbf{x})$. Suppose that $\alpha$ and $\beta$ are not both equal to 0. Since $F|_{V^d_{k,s} \cup V^d_{k,s+1}} = 0$, the line $\alpha x_1 + \beta x_2 = 0$ passes through the points $\phi_{1,2}(V^d_{k,s})$ and $\phi_{1,2}(V^d_{k,s+1})$, where $\phi_{1,2}(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}))$. This is impossible because the line that passes through these points is uniquely defined and does not contain 0. This contradiction proves our claim.

Next we prove that all diagonal elements of the form $f$ are equal to 0. Suppose that one of them is nonzero. Without a limitation of generality we may assume that $t = f_{11} \neq 0$. Let $F$ be the symmetrization of $f$ by the group $G = St(1)$ of permutations which leave 1 fixed. The matrix of $F$ is

$$\begin{bmatrix} t & \beta & \beta & . & . & \beta \\ \beta & \alpha & \delta & . & . & \delta \\ \beta & \delta & \alpha & \delta & . & \delta \\ \ldots & & & & & \\ \beta & \delta & . & \delta & \alpha & \delta \\ \beta & \delta & . & . & \delta & \alpha \end{bmatrix} \tag{11}$$

Consider the following vectors: $v_1 = [1^s, 0^{d-s}] - (s-1)\frac{\mathbf{j}}{n} \in V^d_{k,s}$, $v_2 = [1^{s+1}, 0^{d-s-1}] - s\frac{\mathbf{j}}{n} \in V^d_{k,s+1}$. The conditions $F(v_1) = 0$, $F(v_2) = 0$ yield

$$t + 2(s-1)\beta + (s-1)\alpha + (s-1)(s-2)\delta - \frac{2(s-1)(t + (s+d-2)\beta + (s-1)\alpha + (s-1)(d-2)\delta)}{n} + \frac{(s-1)^2(t + 2(d-1)\beta + (d-1)\alpha + (d-1)(d-2)\delta)}{n^2} = 0, \quad (12)$$

$$t + 2s\beta + s\alpha + s(s-1)\delta - \frac{2s(t + (s+d-1)\beta + s\alpha + s(d-2)\delta)}{n} + \frac{s^2(t + 2(d-1)\beta + (d-1)\alpha + (d-1)(d-2)\delta)}{n^2} = 0. \quad (13)$$

We also have the conditions that the sums of the diagonal and the off-diagonal elements are equal to 0:

$$t + (d-1)\alpha = 0, (d-1)(d-2)\delta + 2(d-1)\beta = 0. \quad (14)$$

The determinant of the system of these 4 equations in variables $\alpha$, $\beta$, $\delta$ and $t$ is equal to

$$\frac{2}{n}(d-1)(s-d+1)(s-d)(d-2-n) \quad (15)$$

and it is not equal to 0 for the specified values of $d$, $n$ and $s$. Hence $t = 0$.

Next we prove that all off-diagonal elements of $f$ are equal to 0. Without a limitation of generality we can assume that $f_{12} \neq 0$. Let $F$ be the symmetrization of $f$ by the group of permutations which map the set $\{1, 2\}$ onto itself. The matrix of $F$ is

$$\begin{bmatrix} t & \beta & \beta & . & . & \beta \\ \beta & \alpha & \delta & . & . & \delta \\ \beta & \delta & \alpha & \delta & . & \delta \\ \dots & & & & & \\ \beta & \delta & . & \delta & \alpha & \delta \\ \beta & \delta & . & . & \delta & \alpha \end{bmatrix}. \quad (16)$$

where $\alpha \neq 0$. We consider the following vectors: $v_1 = [1^{s+1}, 0^{d-s-1}] - s\frac{\mathbf{j}}{n} \in V^d_{k,s}$ and $v_2 = [0^{d-s-1}, 1^{s+1}] - s\frac{\mathbf{j}}{n} \in V^d_{k,s+1}$. The equations $F(v_1) = 0$, $F(v_2) = 0$ yield

$$4(s-1)\beta+2\alpha+(s-1)(s-2)\delta-2\frac{s}{n}(2\alpha+2(s-1+d-2)\beta+(s-1)(d-3)\delta)+ \frac{(s-1)^2(2\alpha+4(d-2)\beta+(d-2)(d-3)\delta)}{n^2} = 0, \quad (17)$$

$$(s+1)s\delta-\frac{2(s+1)s(2\beta+(d-3)\delta)}{n}+\frac{s^2(2\alpha+4(d-2)\beta+(d-2)(d-3)\delta)}{n^2} = 0. \quad (18)$$

We also know that the sum of off-diagonal elements of the matrix of $F$ is equal to 0:

$$2\alpha + 4(d-2)\beta + (d-2)(d-3)\delta = 0 \quad (19)$$

so, with some simplifications, the previous two equations can be rewritten as

$$4(s-1)\beta+2\alpha+(s-1)(s-2)\delta-2\frac{s}{n}(2\alpha+2(s+d-3)\beta+(s-1)(d-3)\delta) = 0,$$
$$\delta - \frac{2(2\beta+(d-3)\delta)}{n} = 0. \quad (20)$$

The systems of equations (19) and (20) in variables $\alpha$, $\beta$ and $\delta$ has determinant $8(d-2-n)(-d+s+1)$ which is not equal to 0 for the specified parameters $d$, $n$ and $s$ which proves that the system has only a 0 solution. In particular, $\alpha = 0$. We have proven that all off-diagonal elements of the matrix of form $f$ are equal to 0. $\square$

# 6 Proof of Theorem 5

Assume that $D \subset \mathbb{R}^{d+1}$, with $0 \notin \operatorname{aff} D$. Consider the lattice $\Lambda =< \operatorname{vert} D >_{\mathbb{Z}}$. Let $v \in \operatorname{vert} D$ be any vertex. Consider polytope $D' = v - D$. We have $\operatorname{vert} D' \subset \Lambda$. One can see that there is a positive definite quadratic form $\phi$ whose ellipsoid circumscribes $\operatorname{vert} D \cap \operatorname{vert} D'$ and does not have any other lattice points inside or on the boundary. Note that since $D$ satisfies the perfection property, $\phi$ is defined up to a parameter and a scale factor.

We can now "stretch" the ellipsoid until it touches some other lattice point(s) $X$. If the vertices of $D$ cannot be embedded into the set of vertices of a centrally symmetric perfect Delaunay polytope, then the ellipsoid will indeed meet points in lattice $\Lambda$. Proving that means proving that the cylinder which circumscribes both polytopes $D$ and $D'$ contains other points of $\Lambda$. If that is not true, project $\Lambda$ onto $\operatorname{lin} D'$ along $t$. The image $\Lambda_1$ is either a lattice, in which case the ellipsoid which circumscribes $D'$ will circumscribe a centrally symmetric perfect Delaunay polytope in $\Lambda_1$, or the direct sum of a lattice and a linear subspace, in which case the set of vertices of $D'$ has dimension less than $d$-a contradiction.

The resulting ellipsoid circumscribes the polytope $D'' = \operatorname{conv}(\operatorname{vert} D \cup \operatorname{vert} D' \cup X)$ and has no other lattice points inside or on the boundary. Hence $D''$ satisfies the Delaunay property. It is easy to see that it also satisfies the perfectness property.

# 7 Summary

This paper has presented two equivalent definitions of perfect Delaunay polytope, one related to the study of edge forms in L-type domains, the other to a generalization of perfect forms to inhomogeneous case. Edge forms classification is one of the cornerstone problems in geometry of positive quadratic forms; as Voronoi noticed in 1908 , the study of minima of inhomogeneous forms is strongly related to the study of perfect homogeneous quadratic forms.

We observed that each centrally perfect Delaunay polytope can be embedded into a centrally symmetric one. Therefore centrally symmetric perfect Delaunay polytopes play a special role. We conjectured that the growth of the number of perfect Delaunay polytopes in dimension $d$ up to affine equivalence is not prohibitive; M. Dutour (2004) discovered that there is only one perfect Delaunay polytope in dimension 6. We have given a new construction of infinite series of perfect Delaunay polytopes in dimensions $d \geq 6$, based on the Gosset polytope $3_{21}$.

Further study of perfect Delaunay polytopes can have two directions: one to estimate the growth of number of types of centrally symmetric Delaunay supertopes, the other to find higher-dimensional analogues of the 15, 16 and 22, 23-dimensional examples.

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